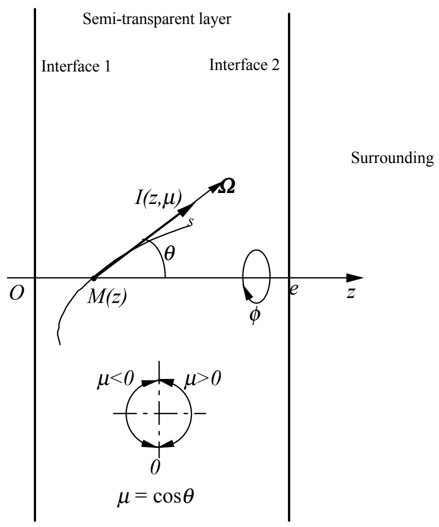
A Conduction-Radiation Coupled Model in Transitory Regime: The Semi-Transparent "Walls"

Alain DEGIOVANNI and Benjamin REMY
Laboratoire d'Energétique et de Mécanique Théorique et Appliquée
L.E.M.T.A – U.M.R C.N.R.S 7563 / E.E.I.G.M – E.N.S.E.M
Université Henri Poincaré-Nancy I, Institut National Polytechnique de Lorraine (I.N.P.L),
02, avenue de la Forêt de Haye – B.P 160, 54 516 Vandoeuvre-Lès-Nancy Cedex, France

A method that allows to solve in a semi-analytical manner the problem of conductive and radiative transfer in a plate of glass is presented. The semi-transparent layer emits, absorbs and scatters radiation (participating medium). This method stems from the principle of a Kernel substitution technique applied to the radiative flux expression and initially introduced by *Lick* (1963) that allows to change the character of the governing heat equation from the integro-differential form to a purely differential one. In the case of limiting cases of purely scattering and purely absorbing media, the solution of the radiative transfer equation is exact. In the general case, we make a two-flux approximation. In all cases, we assume a linear transfer and use the Laplace transform. The advantage of the method is fast computational times for good precision.

Transient Heat Transfers in a Semi-transparent Material (S.T.M)

The general transient heat transfer in a semi-transparent material (*S.T.M)* is obtained by solving the combined conductive (*Fourier*'s law) and radiative (*R.T.E*) equations that are given by :



$$\lambda \Delta T - div(\vec{q}_r) = \rho C_p \frac{\partial T}{\partial t} \quad \textit{Fourier's Law} \qquad (1)$$

$$\frac{dI'_\nu}{ds} + \beta_\nu I'_\nu = \beta_\nu S_\nu \qquad R.T.E \qquad (2)$$

with:

$$S_\nu = (1-\omega_\nu) I_\nu{}^0(T) + \frac{\omega_\nu}{4\pi} \int_0^{4\pi} p_\nu(\Delta' \to \Delta) I'_\nu(s,\Delta') d\Omega'$$

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial z}\frac{\partial z}{\partial s} = \cos(\theta)\frac{\partial}{\partial z} = \mu \frac{\partial}{\partial z}$$

Figure 1 : *Radiative Heat Transfer for a Plane-Parallel Medium*

The temperature is coupled with the intensity through the divergence of the radiative flux that appears in the heat equation as a source term. The intensity is a function of the temperature field through the emission term of the *R.T.E* that corresponds to the Blackbody's intensity which is a function of the local temperature of the material.

The second-left term of the *R.T.E* is the loss of the radiant energy due to the absorption and scattering of the incident radiation : $\beta_v = K_v + \sigma_v$, K_v and σ_v are respectively the monochromatic extinction, absorption and scattering coefficients. $\omega_v = \sigma_v / \beta_v$ denotes the monochromatic albedo and $I_v'^0(T)$, the *Planck*'s energy distribution. The right term of the *R.T.E* is the source function that represents the gain in radiant energy by emission and incoming scattering. $p_v(\Delta' \to \Delta)$ is the phase function.

Solution in the Case of a Grey Plane Parallel Medium

In this section, we assume the radiative properties of the material as being independent of wavelength (hypothesis of a grey medium). This assumption is not necessary and the calculation can also be conducted in the cases of a non-grey or grey-band models. It only allows to obtain more simplified expressions $(I_v' \to I')$. In the more general case, this equation remains difficult to solve. Nevertheless, the *R.T.E* is greatly simplified if the one-dimensional geometry is considered.

One more assumption consists in admitting azimuthal symmetric radiation (intensity is independent of the angle φ in the medium). This is valid for some particular radiative boundary conditions. Due to the singularity of the intensity that appears in the *R.T.*E for $\mu = 0$ and to allow the writing of the boundary conditions in a convenient way, intensity is separated into a forward component $I^+(z,\mu)$ for $\mu > 0$ and a backward component $I^-(z,\mu)$ for $\mu < 0$. The *R.T.E* is then split into two coupled integrodifferential equations that have explicitly the same forms:

$$\frac{\mu}{\beta}\frac{dI'^{+/-}(z,\mu)}{dz} + I'^{+/-}(z,\mu) = (1-\omega)I^0(T) + \mathrm{K}$$

$$\mathrm{K} + \frac{\omega}{2}\left[\int_0^1 p(\mu,\mu')I'^{+/-}(z,\mu')d\mu' + \int_{-1}^0 p(\mu,\mu')I'^{+/-}(z,\mu')d\mu'\right] \quad (3)$$

The integrodifferential equations can be solved by introducing the radiative conditions of he problem. In the case of opaque, diffusely emitting (ε independent of the angle) and diffusely reflecting boundaries (ρ independent of the angle), the intensities of the surfaces no longer depends on θ and can be written under the following form:

$$I^+(0) = \varepsilon_1 I^0(T_1) + 2\rho_1 \int_0^1 I^-(0,-\mu')\mu' d\mu' \quad \text{and} \quad I^-(e) = \varepsilon_2 I^0(T_2) + 2\rho_2 \int_0^1 I^+(e,\mu')\mu' d\mu' \quad (4)$$

Solution of the Radiative Transfer Equation : *" Equivalent Radiative Resistance "*

First attempts to solve this combined equations consists in modeling the radiative transfer by an equivalent radiative resistance. In the case of optically thin material, we can make the assumption of the Local Thermodynamic Equilibrium (*L.T.E*), which yields to a flux divergence equal to zero. In the case of a optically thick medium, the radiative transfer can be viewed like a pure diffusion process (*Rosseland*'s model). This notion has been then extended to intermediate optical thickness by several authors.

In all theses cases, the radiative transfer is uncoupled with the temperature field within the material but remains linked to the conductive flux through the boundaries conditions. A very simple model can be purposed to determine the transient heat transfer in the *S.T.M:* the conductive heat transfer is modelled through a quadrupole formulation[1] that allows to linearly links by a transfer matrix the *"inner"* and *"outer" Laplace* temperatures and total fluxes transforms:

$$\begin{bmatrix}\bar{\theta}(0)\\ \bar{\phi}(0)\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\bar{\theta}(1)\\ \bar{\phi}(1)\end{bmatrix} \tag{5}$$

The radiative transfer is taken into account by a radiative resistance in parallel with the pure conductive quadrupole. The more interesting resistance expression is given by the *Poltz-Jugel*'s model[2] that is valid on a wide range of optical thickness values:

$$\textit{Poltz-Jugel's Model}: \ (\tau_0 \sim 1) \qquad \lambda_r = \frac{16}{3} n^2 \sigma \frac{T_{ext}^{\ 3}}{\beta} Y \tag{6}$$

$$\text{with: } Y = 1 - \frac{3}{4\tau_0}\left(1-4E_5(\tau_0)\right) - \frac{2}{3\tau_0}\frac{\left(1-(\varepsilon_1+\varepsilon_2)/2-2\rho_1\rho_2E_3(\tau_0)\right)}{1-4\rho_1\rho_2E_3(\tau_0)^2}\cdot\left(1-3E_4(\tau_0)\right)$$

This model tends to the *Deissler's* model[3] for low optical thicknesses and to the well-known *Rosseland's* model for large optical thicknesses $(Y \to 1)$.

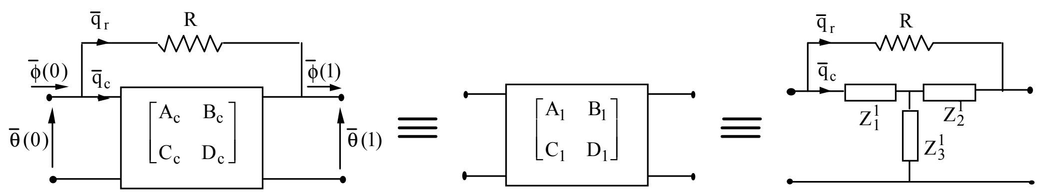


Figure 2 : *Equivalent Radiative Resistance*

In the particular case of a pure scattering material, the emission term disappears in the *R.T.E.* The radiative and conductive equations are naturally uncoupled and the model based on the notion of a radiative resistance is exact. This explains the success of this notion in insulating materials like foams.

The Semi-Tranparent Quadrupole

The general equation for the combined conductive and radiative non-steady heat transfer is an integrodifferential equation that can be solved analytically by making the assumption of a linearized radiative transfer $(1+\theta_0/T_0)^4 \simeq 1+4\theta_0/T_0$ (T_0 being the reference temperature). The semi-transparent material is always considered in small deviations from a reference state. In the case of a purely absorbing-emitting medium, the intensity is anisotropic and the problem is difficult to solve because of the presence of exponential integral functions $E_n(x)=\int_0^1 e^{-x/\mu}\mu^{n-2}d\mu$ in the radiative flux expression.

Applying a Kernel substitution technique that consists in replacing exponential integrals that appear in the radiative flux expression, we obtain a modified integral equation. Here, we used the approximate kernel found by *Lick*[4] of the form $a\exp(-bz)$ for the correct kernel $E_2(z)$, and also $(a/b)\exp(-bz)$ for the Kernel $E_3(z)$, with a and b being equal to $3/4$ and $3/2$ respectively. By using the coefficients a and b derived by *Lick*, we obtain discrepancies less than 3% on the temperature field. The modified integral equation is given in dimensionless form by:

$$q_r(z)=I^+(0)e^{-\tau z}-I^-e^{-\tau(1-z)}-\frac{1}{4}\left(e^{-\tau z}-e^{-\tau(1-z)}\right)$$
$$+\frac{\tau}{T_0}\int_0^z\theta(z')\exp(-\tau(z-z'))dz'-\frac{\tau}{T_0}\int_z^1\theta(z')\exp(-\tau(z'-z))dz' \quad \text{with: } \tau=\tfrac{3}{2}\tau_0 \qquad (7)$$

We have to notice that in the case of the two-flux approximation, the Kernel substitution technique is not required because classical exponential expressions naturally appears in the expression of the radiative flux. Then, it is possible to obtain from expression (7), a differential equation in terms of the radiative flux[5]:

$$\frac{\partial^2 q_r}{\partial z^2}=\tau^2 q_r+2\frac{\tau}{T_0}\frac{\partial\theta}{\partial z} \qquad (8)$$

Differentiating the energy equation twice, substituting the second-order derivative of the radiative flux and expressing $\partial q_r/\partial z$ in term of the partial derivatives of θ according to the energy equation yields to:

$$\frac{\partial^4\theta}{\partial z^4}-\frac{\partial^3\theta}{\partial z^2\partial t}-2\frac{\tau^2}{N}\frac{\partial^2\theta}{\partial z^2}-\tau^2\frac{\partial^2\theta}{\partial z^2}+\tau^2\frac{\partial\theta}{\partial t}=0 \qquad (9)$$

If we now introduce the *Laplace* temperature transform $\bar{\theta}(p)=\int_0^\infty\theta(t)\exp(-pt)dt$ and applying the integral transformation to the above equation leads to the following differential equation:

$$\frac{d^4\bar{\theta}}{dz^4}-\left(p+2\frac{\tau^2}{N}+\tau^2\right)\frac{d^2\bar{\theta}}{dz^2}+p\tau^2\bar{\theta}=0 \quad\rightarrow\quad a\frac{d^4\bar{\theta}}{dz^4}+b\frac{d^2\bar{\theta}}{dz^2}+c\bar{\theta}=0 \qquad (10)$$

The solution of the preceding fourth-order ordinary equation gives the *Laplace* transform of the temperature $\bar{\theta}(z)=\sum_{i=1}^{4}\alpha_i\exp(\gamma_i z)$ with : $\gamma_{1,2,3,4}=\pm\left[\left(-b\pm\sqrt{b^2-4ac}\right)/2a\right]^{1/2}$. (11)

The *Laplace* transform of the total flux $\bar{\phi}$ (sum of the conductive and radiative fluxes) is given by: $\bar{\phi}=-\frac{d\bar{\theta}}{dz}+\frac{\tau T_0}{N}\bar{q}_r$. (12)

The matrix coefficients A, B, C and D are obtained by writing the relations between $\bar{\theta}(0)$, $\bar{\theta}(1)$, $\bar{\phi}(0)$ and $\bar{\phi}(1)$:

$$\bar{\theta}(0)=\sum_{i=1}^{4}\alpha_i\ ,\ \bar{\theta}(1)=\sum_{i=1}^{4}\alpha_i \exp(\gamma_i),\ \bar{\phi}(0)=\sum_{i=1}^{4}\alpha_i\gamma_i\delta_i \quad \text{and} \quad \bar{\phi}(1)=\sum_{i=1}^{4}\alpha_i\gamma_i\delta_i \exp(\gamma_i) \qquad (13)$$

Two other relations between the α_i's can be obtained by requiring that the solution has to satisfy the heat transfer equation (compatibility's equations):

$$\sum_{i=1}^{4} X_i\alpha_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} Y_i\alpha_i = 0 \qquad (14)$$

So, we have:

$$\begin{bmatrix}\bar{\theta}(1)\\ \bar{\phi}(1)\\ 0\\ 0\end{bmatrix}=\begin{bmatrix} e^{\gamma_1} & e^{\gamma_2} & e^{\gamma_3} & e^{\gamma_4}\\ \delta_1\gamma_1 e^{\gamma_1} & \delta_2\gamma_2 e^{\gamma_2} & \delta_3\gamma_3 e^{\gamma_3} & \delta_4\gamma_4 e^{\gamma_4}\\ X_1 & X_2 & X_3 & X_4\\ Y_1 & Y_2 & Y_3 & Y_4\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} \text{and} \begin{bmatrix}\bar{\theta}(0)\\ \bar{\phi}(0)\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\ \delta_1\gamma_1 & \delta_2\gamma_2 & \delta_3\gamma_3 & \delta_4\gamma_4\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} \qquad (15)$$

That can be formally written as $I = NM^{-1}O$. The coefficients of the quadrupole are obtained by extracting the two first columns and lines from the matrix NM^{-1}.

Conclusions

The coupled conducto-radiative quadrupole we purposed in this paper allows to solve the heat transfer within a Semi-Transparent Wall (*M.S.T*) in a very simple way. The main interest of this method is that it is non-dependant of the boundaries conditions and it can be applied without any major difficulties to any kind of imposed temperatures or fluxes problems (heat pulse[6], step of temperature, and so on ...). The steady-state solution can be easily obtained by taking the following limit $\lim_{p\to 0} p\bar{\theta}(p)$. This model can also be extended to systems in a periodic regime by setting $p = i\omega$.

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[5] Cogley J.C., Vincenti W.G. and Gilles S.E., AIAA J. **3**, p. 155-553 (1968).

[6] André S. and Degiovanni A., Int. J. Heat & Mass Transfer **38**, p. 3401-3412 (1995).