A Coupled Conductive and Radiative Model for Solving the Steady-State and Transient Heat Transfers in Semi-Transparent " Walls "

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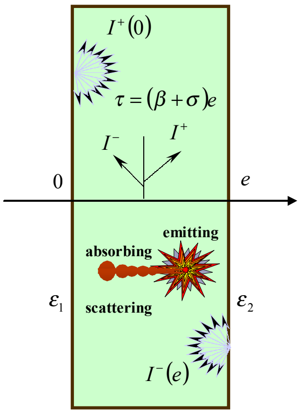
A method that allows to solve in a semi-analytical manner the problem of conductive and radiative transfer in a semi-transparent material that emits, absorbs and scatters radiation (participating medium), is presented. The major principles and assumptions that allow to built the analytical model are exposed and its representation under a quadrupole or matrix formulation is presented. Based on the use of this semi-analytical model, several applications are considered in steady-state, transient and modulated regimes. The advantage of this method is fast computational times for good precision.

Coupled Heat Transfers in a Semi-transparent Material (*S.T.M*)

The general transient heat transfer within a semi-transparent material is obtained by solving the combined conductive (*Fourier*'s law) and radiative (*R.T.E*) equations. The temperature is coupled with the intensity through the divergence of the radiative flux that appears in the heat equation as a source term. The intensity is a function of the temperature field through the emission term of the *R.T.E*.



$$\lambda \Delta T - div(\vec{\varphi}_r) = \rho C_p \frac{\partial T}{\partial t} \quad (1)$$

Fourier's Law

$$\frac{dI'_\nu}{ds} + (K_\nu + \sigma_\nu) I'_\nu = K_\nu I_\nu^0(T) + \frac{\sigma_\nu}{4\pi} \int_0^{4\pi} p_\nu(\Delta' \to \Delta) I'_\nu(\Delta', s) d\Omega' \quad (2)$$

R.T.E.

$p_\nu(\Delta' \to \Delta)$: *Phase Function*

Figure 1 : *Combined Heat Transfers in a Semi-Transparent Material (S.T.M)*

The second-left term of the *R.T.E* is the loss of the radiant energy due to the absorption of the incident radiation (K_ν and σ_ν are the monochromatic absorption and scattering coefficients respectively). The right terms of the *R.T.E* represent the gain in radiant energy due to emission and incoming diffusion effect.

Solution of the Radiative Transfer Equation (*R.T.E*)

In the case of heat transfer between two parallel planes with an infinite expansion, an analytical solution of the coupled conductive-radiative equations can be provided under the assumption of a linear heat transfer (small temperature variations). Approximating the exponential integral functions $E_n(x)$, that naturally appear in the radiative flux expression, by pure exponentials, the radiative expression that allows to take into account the radiative transfer between the boundaries and the bulk exhibits an intrinsic property that allows to provide an analytical solution of the combined heat transfers[1].

Solution in transient regime

Taking the *Laplace* transform of the transient heat transfer equation, differentiating twice this equation and applying the modified expression of the radiative flux, leads to a pure differential equation for the temperature in the *Laplace* domain $\overline{\theta}(p)$,

$$\lambda \frac{\partial^2 T}{\partial z^2} - \frac{\partial q_r}{\partial z} = \rho C_p \frac{\partial T}{\partial z} \quad \rightarrow \quad \frac{d^4\overline{\theta}}{dz^4} - \left(p + \frac{2\tau^2}{N} + \tau^2 \right) \frac{d^2\overline{\theta}}{dz^2} + p\tau^2 \overline{\theta} = 0 \tag{3}$$

The solution is simply $\overline{\theta}(z) = \sum_{i=1}^{4} A_i \exp(\gamma_i z)$. The four integration constants A_i can be obtained by introducing the two thermal boundary conditions and requiring that the solution must satisfy the *Laplace* transform of the modified heat equation (2 conditions).

Solution in steady- state regime

The same method in steady-state regime leads to the following solution:

$$\frac{d^4\theta}{dz^4} - \left(\frac{2\tau^2}{N} + \tau^2 \right) \frac{d^2\theta}{dz^2} = 0 \quad \rightarrow \quad \theta(z) = Az + B + (C/\omega^2)\mathrm{e}^{\omega z} + (D/\omega^2)\mathrm{e}^{-\omega z} \tag{4}$$

Steady-State Temperature field in a vitroceramic

In drying technologies, infra-red panels are developed that are composed of a cavity, containing the infra-red power generator (quartz tubes), closed by a vitroceramic window, that protects the infra-red lamps from water ejections. The detailed procedure of the modeling has been exposed by V. Manias[2]. The interfaces are considered semi-transparent and the non-gray character of the ceramic glass is considered through a multi-band analysis. In steady-state, the governing equation is: $\frac{\partial^2\theta}{\partial z^2} = \frac{1}{k} \partial \left(\sum_{i=1}^{n} q_{r_{\Delta\lambda_i}} \right) \Big/ \partial z$.

With the band expression of the radiative flux, the same method of resolution as above can be applied and coupled to a method of radiosity for surface radiative exchanges and to a numerical code of fluid mechanics (*ESTET*) for solving the convective problem in the cavities, the *3-D* problem of heat transfer in a cavity divided into two parts by a window glass has been solved. **Figure 2** gives temperature profiles obtained in the non-gray case for different band discretizations of the absorption spectrum of vitroceram.

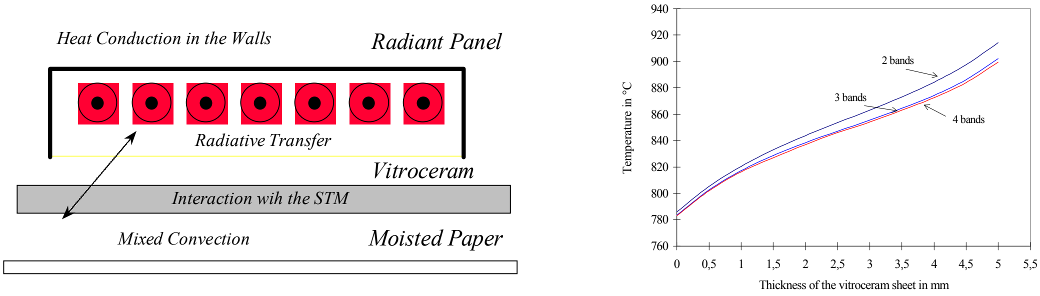


Figure 2 : *Effect of the Non-GrayBand Discretization on the Temperature Profile*

The Semi-Tranparent Quadrupole

The quadrupole formulation[3] can be efficiently used for the determination of transient heat transfers in a *S.T.M*. This method allows to linearly link the inner and outer values (temperatures an fluxes) through a transfer matrix. The coefficients of this matrix can be obtained from the analytical solution previously presented (see relation (3)).

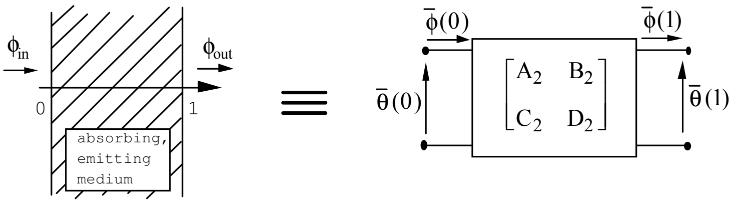


Figure 3 : *The Semi-Transparent Quadrupole (S.T-Quad)*

The main advantage of this technique is that the solution is non-dependant of the boundaries conditions. The *Laplace* inversion is implemented by a numerical algorithm[4]. The advantage of this method stays in its low time computer consuming and good accuracy that allows to characterise by an inverse technique the thermophysical properties of a *S.T.M* from experimental data[5]. See for instance, the measurement of the phononic diffusivity by a *Flash* method. The accuracy of the solution has been numerically verified in the case of the temperature response of the excited side of a glass sample subjected to a pulse and step irradiations and in the presence of heat loss

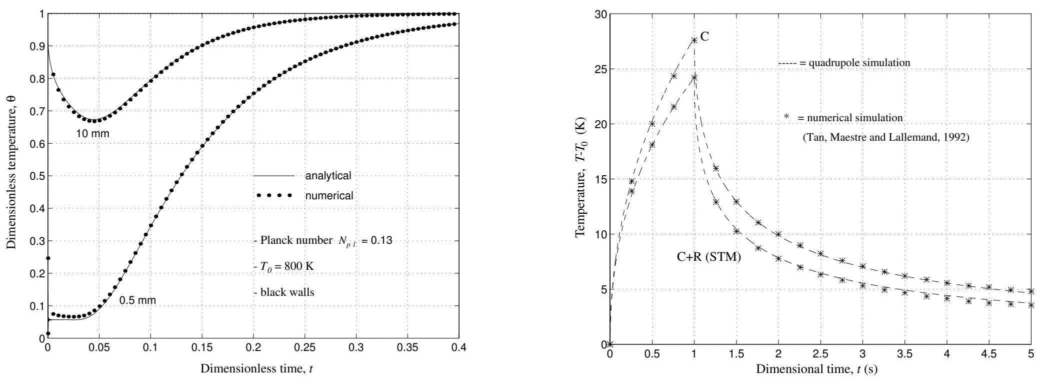
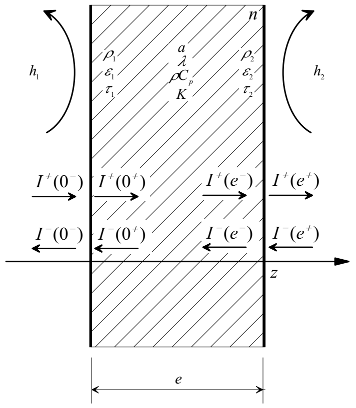


Figure 4 : *Validation of the S.T-Quad in the Cases of a Pulse and Step Irradiations*

The Matrix Formulation : *Multilayered S.T.M / Anisotropic Scattering*

The main limitation of the quadrupole formulation is that the boundaries have to be opaque. So, it is not possible to solve the heat transfers in a multilayered *S.T.M.* The difficulty is that the radiative flux in a given layer depends of the fluxes in the next layers. This problem can be solved by using a matrix formulation of the solution. Indeed, the temperature and flux can be fully determined if we know the values of A_i's, $I^+(0)$, $I^-(0)$, $I^+(e)$ and $I^-(e)$ that can be obtained from 8 different relations or equations.



$$
\begin{bmatrix}
\pi & -\pi & M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & 0 & 0 \\
1 & -\rho_1 & M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} & 0 & 0 \\
1 & 0 & M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} & 0 & 0 \\
-e^{-K'e} & 0 & M_{4,1} & M_{4,2} & M_{4,3} & M_{4,4} & 1 & 0 \\
0 & 1 & M_{5,1} & M_{5,2} & M_{5,3} & M_{5,4} & 0 & -e^{-K'e} \\
0 & 0 & M_{6,1} & M_{6,2} & M_{6,3} & M_{6,4} & 0 & 1 \\
0 & 0 & M_{7,1} & M_{7,2} & M_{7,3} & M_{7,4} & -\rho_2 & 1 \\
0 & 0 & M_{8,1} & M_{8,2} & M_{8,3} & M_{8,4} & \pi & -\pi
\end{bmatrix}
\begin{pmatrix}
\bar{L}^+(0) \\ \bar{L}^-(0) \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \bar{L}^+(e) \\ \bar{L}^-(e)
\end{pmatrix}
=
\begin{pmatrix}
\bar{\varphi}_0 \\ \varepsilon_1\alpha \\ \alpha \\ \alpha\left(1-e^{-K'e}\right) \\ \alpha\left(1-e^{-K'e}\right) \\ \alpha \\ \varepsilon_2\alpha \\ 0
\end{pmatrix}
$$

Matrix Formulation in a case of one S.T.M layer submitted to a heat pulse

Figure 5 : *Matrix Formulation of a S.T.M*

First and last lines correspond to the conductive-radiative boundary conditions, second and seventh to the pure radiative conditions, third and sixth to the compatibility's relations and fourth and fifth to the relations of the intensities between the surfaces.

The heat transfer in a bilayer material can be easily solved by writing such a matrix in each layer of the material. This method is especially of a great interest for the characterization of non free-standing synthetic diamond films deposited by a chemical process (*C.V.D* technique) on a silicon substrate and to take into account the strong anisotropy of the material in its thickness[7]. Such a method has also been used to model the anisotropic scattering effects[8] that appear in particular insulating materials like foams. The calculation are a little bit more complicated but yields to a similar matrix formulation.

A Periodic Method for Glass Characterization

Such a technique has also been used in a modulated regime for characterization of Glass properties[9]. The analytical model of the *S.T.M* remains strictly the same. (*Laplace* parameter p has only to be replaced by $j\omega$). The *Bode* diagram of the complex temperature θ_r^* versus dimensionless frequency $f^* = f\, t_c$ are presented in **Figure 6**. The curves labeled *'0'* are those obtained for the pure conductive case (opaque sample). The curves labeled *'1','2','3'* and *'4'* have been obtained for different optical thickness τ and *Planck* number N_{pl} values respectively equal to (0.5,0.5),(5,0.5), (0.5,5), (5,5) and in the case of black boundaries. The strong diversity of these curves and the great variety of phase lag thermogram allow the experimentalist to identify approximate values for parameter τ (degree of semi-transparency) and parameter N_{pl} (conductive-to-radiative ratio) that are usually difficult to obtain by a *Flash* method.

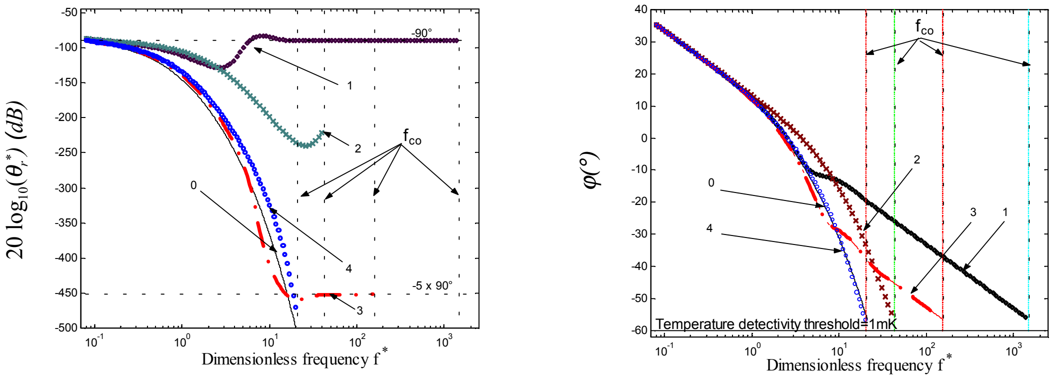


Figure 6 : *Amplitude and Phase in a S.T.M versus Reduced Frequency*

Conclusions

The semi-analytical model we presented in this paper is an efficient method to solve numerous coupled problems in steady-state, transient and modulated regimes. The main restrictive assumption is the linearization that is required to obtain an analytical solution. For systems with large temperature gradients, it is always possible to split the *S.T.M* into several elements in which this linearization is valid. Even in this case, this model remains more accurate than a P_1 / P_3-*F.E.M's* model because the interpolation functions (exponential functions) we used in each element are better to fit the solution than polynomial functions. The main advantage of this method also stays in its fast computational times and accuracy that makes this model very convenient in control applications and in inverse problems.

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